

On the variety of two dimensional real associative algebras

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Abstract

This paper consists of a description of the variety of two dimensional associative algebras within the framework of Nonstandard Analysis. By decomposing each algebra in A^2 as sum of a Jordan algebra and a Lie algebra, we calculate the isomorphism classes of two dimensional associative algebras over the field of real numbers and determine the open components and the contractions of the variety.

Mathematics Subject Classification: 16B99

Keywords: associative algebras, rigidity, contraction

¹This author has been partially supported by the research project MTM2006-09152 of Ministerio de Educación y Ciencia

1 Introduction

The aim of this work is to study some properties of the variety of two dimensional associative algebras, specially those concerning rigidity and contractions. We first obtain a decomposition of associative algebras as sum of a Jordan algebra and a Lie algebra, which enables us to use known results on Jordan algebras [1] to classify two dimensional associative algebras over the field of real numbers. Then we introduce the concept of perturbation within Nelson's Internal Set Theory [6] and derive the perturbation equations of the variety. Nonstandard Analysis tools permit us to prove that \mathcal{A}^2 has four open components, two of dimension 4 and two of dimension 2. The remaining algebras of the variety are obtained by contraction of the rigid algebras which define the open components.

Definition 1. *An associative algebra law over \mathbb{R} is a bilinear mapping $\beta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying the constraint*

$$\beta(\beta(x, y), z) - \beta(x, \beta(y, z)) = 0, \quad x, y, z \in \mathbb{R}^n. \quad (1)$$

We will abbreviate $\beta(x, y)$ by $x \circ y$, and \mathcal{A}^n will denote the set of associative algebras over \mathbb{R}^n . The ordered pair (\mathbb{R}^n, β) is called associative algebra.

Definition 2. *A Jordan algebra law over \mathbb{R} is a symmetric bilinear mapping $\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which verifies the identity*

$$\varphi(\varphi(x, x), \varphi(x, y)) - \varphi(x, \varphi(\varphi(x, x), y)) = 0, \quad x, y \in \mathbb{R}^n.$$

The ordered pair (\mathbb{R}^n, φ) is a Jordan algebra and \mathcal{J}^n will denote the set of Jordan algebra laws.

Definition 3. *A Lie algebra law over \mathbb{R}^n is an alternate bilinear mapping $\mu : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying the Jacobi identity*

$$\mu(\mu(x, y), z) + \mu(\mu(y, z), x) + \mu(\mu(z, x), y) = 0, \quad x, y, z \in \mathbb{R}^n.$$

We will denote the set of Lie algebras over \mathbb{R}^n by \mathcal{L}^n . The ordered pair (\mathbb{R}^n, μ) , where $\mu \in \mathcal{L}^n$, is a Lie algebra.

Now we enunciate a proposition which plays a fundamental role in the classification which will be given.

Proposition 1. *Let (\mathbb{R}^n, \circ) be a real associative algebra. Then:*

1. *The law φ defined by*

$$\varphi(x, y) = \frac{x \circ y + y \circ x}{2}, \quad x, y \in \mathbb{R}^n$$

is a Jordan algebra law.

2. The law μ defined by

$$\mu(x, y) = \frac{x \circ y - y \circ x}{2}, \quad x, y \in \mathbb{R}^n$$

is a Lie algebra law.

Let us recall that if there exists a non-zero vector u such that $\beta(u, v) = 0$ for all $v \in \mathbb{R}^n$, we say that β has left isotropy. Similarly, if $\beta(v, u) = 0$ for all $v \in \mathbb{R}^n$, then u is a right isotropic vector. An associative algebra is said to be simple if it does not admit any proper ideal.

Let $B = \{e_1, \dots, e_n\}$ be a basis for \mathbb{R}^n . It is possible to identify each algebra in \mathcal{A}^n with its structure constants, that is, to consider $\beta \in \mathcal{A}^n$ as the tensor $(a_{ij}^k) \in \mathbb{R}^{n^3}$ whose coordinates, univocally determined by $e_i \circ e_j = a_{ij}^k e_k$, are the solutions of the system

$$a_{ij}^h a_{hk}^l = a_{ih}^l a_{jk}^h, \quad 1 \leq i, j, k, h, l \leq n. \quad (2)$$

That gives \mathcal{A}^n a structure of algebraic variety embedded in \mathbb{R}^{n^3} . From now on we will identify each algebra with its law. In the case of associative algebras, non-written products will be supposed to be zero. If φ is a Jordan algebra or a Lie algebra only non-zero products $\varphi(e_i, e_j)$, with $i \leq j$, will be written.

2 Classification of two dimensional real associative algebras

If $n = 2$, real associative algebras are given by the relations

$$\begin{aligned} e_1 \circ e_1 &= a_1 e_1 + a_2 e_2, \\ e_1 \circ e_2 &= b_1 e_1 + b_2 e_2, \\ e_2 \circ e_1 &= c_1 e_1 + c_2 e_2, \\ e_2 \circ e_2 &= d_1 e_1 + d_2 e_2. \end{aligned}$$

Or equivalently by a coefficient matrix of the form:

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}$$

Developing (2) the following equations are obtained:

$$\begin{aligned}
 a_2 b_1 &= a_2 c_1, \\
 a_2 b_2 &= a_2 c_2, \\
 b_1 b_2 &= a_2 d_1, \\
 a_2 d_1 &= c_1 c_2, \\
 a_2 b_1 + b_2^2 &= a_1 b_2 + a_2 d_2, \\
 a_1 c_1 + b_1 c_2 &= a_1 b_1 + b_2 c_1, \\
 a_1 d_1 + b_1 d_2 &= b_1^2 + b_2 d_1, \\
 a_1 c_2 + a_2 d_2 &= a_2 c_1 + c_2^2, \\
 b_1 c_2 + b_2 d_2 &= b_2 c_1 + c_2 d_2, \\
 c_1^2 + c_2 d_1 &= a_1 d_1 + c_1 d_2.
 \end{aligned} \tag{3}$$

Thus, A^2 is an algebraic variety embedded in \mathbb{R}^8 and defined by the above system of homogeneous polynomials.

Let us consider the natural action of the general linear group $GL(n, \mathbb{R})$ over the variety:

$$\begin{aligned}
 GL(n, \mathbb{R}) \times \mathcal{A}^n &\longrightarrow \mathcal{A}^n \\
 (f, \beta) &\longmapsto f^{-1} * \beta * (f \times f),
 \end{aligned}$$

with $f^{-1} * \beta * (f \times f)(x, y) = f^{-1}(\beta(f(x), f(y)))$. For each $\beta \in \mathcal{A}^n$, the orbit under this action, $\mathcal{O}(\beta)$, represents the set of associative algebras isomorphic to β . Our first aim is to determine the space of orbits

$$A^2/GL(2, \mathbb{R}) = \{\mathcal{O}(\beta)\}_{\beta \in \mathcal{A}^2},$$

that is, the isomorphism classes of two dimensional real associative algebras.

For that purpose, we will make use of the decomposition

$$x \circ y = \frac{x \circ y + y \circ x}{2} + \frac{x \circ y - y \circ x}{2} = \varphi(x, y) + \mu(x, y),$$

where φ is a Jordan algebra and μ is a Lie algebra defined by the product $\mu(e_1, e_2) = ae_1 + be_2$. In [1] a classification theorem for \mathcal{J}^2 is proved:

Theorem 1. *Let φ be a non-Abelian two dimensional real Jordan algebra. Then φ is isomorphic to one of the following pairwise non-isomorphic Jordan algebras:*

1. $\varphi_1(e_1, e_1) = e_1, \varphi_1(e_1, e_2) = e_2, \varphi_1(e_2, e_2) = -e_1.$
2. $\varphi_2(e_1, e_1) = e_1, \varphi_2(e_1, e_2) = e_2, \varphi_2(e_2, e_2) = e_1.$
3. $\varphi_3(e_1, e_1) = e_1, \varphi_3(e_1, e_2) = e_2, \varphi_3(e_2, e_2) = 0.$

$$4. \varphi_4(e_1, e_1) = 0, \quad \varphi_4(e_1, e_2) = 0, \quad \varphi_4(e_2, e_2) = e_2.$$

$$5. \varphi_5(e_1, e_1) = e_2, \quad \varphi_5(e_1, e_2) = 0, \quad \varphi_5(e_2, e_2) = 0.$$

$$6. \varphi_6(e_1, e_1) = e_1, \quad \varphi_6(e_1, e_2) = \frac{1}{2}e_2, \quad \varphi_6(e_2, e_2) = 0.$$

Example 1. To illustrate how this result may be applied to solve the classification problem, let us consider the case in which φ is isomorphic to φ_1 . Then, there exists a basis $\{e_1, e_2\}$ such that:

$$e_1 \circ e_1 = \varphi_1(e_1, e_1) + \mu(e_1, e_1) = e_1.$$

$$e_1 \circ e_2 = \varphi_1(e_1, e_2) + \mu(e_1, e_2) = ae_1 + (1+b)e_2.$$

$$e_2 \circ e_1 = \varphi_1(e_2, e_1) + \mu(e_2, e_1) = -ae_1 + (1-b)e_2.$$

$$e_2 \circ e_2 = \varphi_1(e_2, e_2) + \mu(e_2, e_2) = -e_1.$$

Structure constants must satisfy (3), so $a = b = 0$ and β coincides with φ_1 .

Making the same calculus for each $\varphi_i \in \mathcal{J}^2$, the following theorem is obtained. For the first five Jordan algebras we have $a = b = 0$, but when we consider φ isomorphic to φ_6 , the system (3) admits two different solutions: $(0, \frac{1}{2})$ and $(0, -\frac{1}{2})$. Thus, there are seven isomorphism classes in \mathcal{A}^2 .

Theorem 2. Let β be a two dimensional real associative algebra. If β is not Abelian, then β is isomorphic to one of the following pairwise non-isomorphic associative algebras:

$$1. \beta_1 : e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_1 = e_2, \quad e_2 \circ e_2 = -e_1.$$

$$2. \beta_2 : e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_1 = e_2, \quad e_2 \circ e_2 = e_1.$$

$$3. \beta_3 : e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_1 = e_2, \quad e_2 \circ e_2 = 0.$$

$$4. \beta_4 : e_1 \circ e_1 = 0, \quad e_1 \circ e_2 = 0, \quad e_2 \circ e_1 = 0, \quad e_2 \circ e_2 = e_2.$$

$$5. \beta_5 : e_1 \circ e_1 = e_2, \quad e_1 \circ e_2 = 0, \quad e_2 \circ e_1 = 0, \quad e_2 \circ e_2 = 0.$$

$$6. \beta_6 : e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_1 = 0, \quad e_2 \circ e_2 = 0.$$

$$7. \beta_7 : e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = 0, \quad e_2 \circ e_1 = e_2, \quad e_2 \circ e_2 = 0.$$

Moreover, β_2 is the unique simple algebra in \mathcal{A}^2 .

Remark 1. Considered over \mathbb{C} , β_1 and β_2 are isomorphic, with the change of basis given by $x_1 = e_1$ and $x_2 = ie_2$. Thus, we have found an example of a simple associative algebra whose complexification is not simple. In \mathcal{A}^2 , β_3, β_4 and β_5 have right and left isotropy, β_6 has just left isotropy and β_7 has just right isotropy.

Proposition 2. *Let $\beta \in \mathcal{A}^2$ be a two dimensional real associative algebra.*

1. *If β has no isotropy, then β is isomorphic either to β_1 or to β_2 .*
2. *If β has just left isotropy, then β is isomorphic to β_6 .*
3. *If β has just right isotropy, then β is isomorphic to β_7 .*

3 Perturbations of associative algebras

Considering in \mathcal{A}^n the subspace topology induced by \mathbb{R}^{n^3} , it is possible to give the following definition:

Definition 4. *An associative algebra β is rigid in \mathcal{A}^n if its orbit under the action of $GL(n, \mathbb{R})$ is open.*

To study rigidity in the framework of Internal Set Theory, the concept of perturbation is introduced (cf. [2], [4], [6] for the details).

Definition 5. *Let n be standard and let β_0 be a real associative algebra. We say that β is a perturbation of β_0 if $\beta(x, y)$ and $\beta_0(x, y)$ are infinitely close for all x, y standard vectors of \mathbb{R}^n .*

Proposition 3. *A standard associative algebra law $\beta_0 \in \mathcal{A}^n$ is rigid if and only if any perturbation of β_0 is isomorphic to it.*

Proof. If $\beta_0 \in \mathcal{A}^n$ is rigid, then $\mathcal{O}(\beta_0)$ is open. Thus, $\mathcal{O}(\beta_0)$ contains the halo of β_0 and any perturbation β of β_0 is isomorphic to β_0 . Conversely, if any perturbation belongs to $\mathcal{O}(\beta_0)$, then the halo of β_0 is contained in $\mathcal{O}(\beta_0)$ and, by transference, $\mathcal{O}(\beta_0)$ is open, so β_0 is rigid. \square

Now we enunciate a theorem, due to M. Goze [3], basic to determine the rigidity of associative algebras.

Theorem 3. *Let M_0 be a standard point in \mathbb{R}^n . Then every point $M \in \mathbb{R}^n$ infinitely close to M_0 admits a decomposition of the form:*

$$M = M_0 + \epsilon_1 v_1 + \epsilon_1 \epsilon_2 v_2 + \dots + \epsilon_1 \epsilon_2 \dots \epsilon_p v_p,$$

where $\epsilon_1, \epsilon_2, \dots, \epsilon_p$ are infinitely small scalars and v_1, v_2, \dots, v_p linearly independent vectors. Moreover, if $M = M_0 + \eta_1 u_1 + \eta_1 \eta_2 u_2 + \dots + \eta_1 \eta_2 \dots \eta_q u_q$ is another decomposition of M , then $p = q$ and the flag defined by v_1, v_2, \dots, v_p coincides with the flag defined by u_1, u_2, \dots, u_q . The integer p which describes the equivalence class of a point is called length of M .

As a consequence of Goze's theorem, any perturbation of the standard law $\beta_0 \in \mathcal{A}^n$ may be written as

$$\beta = \beta_0 + \epsilon_1 \varphi_1 + \epsilon_1 \epsilon_2 \varphi_2 + \dots + \epsilon_1 \epsilon_2 \dots \epsilon_p \varphi_p, \quad (4)$$

where ϵ_i are infinitely small and $\varphi_i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are independent bilinear mappings. Considering β_0 a standard element of the variety and β a perturbation of β_0 decomposed according to (4), we have that the shade of the straight line which joins β and β_0 is a standard straight line which belongs to the tangent cone to the variety in β_0 .

Let $\beta_0 \in \mathcal{A}^n$ be a standard associative algebra law and let $f \in GL(n, \mathbb{R})$ be standard. For any ϵ infinitely small, the endomorphism $Id + \epsilon f$ belongs to the general linear group, so it makes sense to consider the action

$$(Id + \epsilon f)^{-1} \beta_0 ((Id + \epsilon f), (Id + \epsilon f)) = \beta_0 + \epsilon(\delta_{\beta_0} f) + \epsilon^2(\Delta(\beta_0, f, \epsilon)),$$

where

$$\delta_{\beta} f(x, y) = \beta(f(x), y) + \beta(x, f(y)) - f(\beta(x, y))$$

are the 2-coboundaries of Hochschild cohomology. Moreover, since the straight line which joins β_0 and an infinitely close point β'_0 is tangent to the orbit $\mathcal{O}(\beta_0)$ in β_0 , the tangent space is given by

$$T_{\beta_0} \mathcal{O}(\beta_0) = \{\delta_{\beta_0} f : f \in GL(n, \mathbb{R})\}. \quad (5)$$

The dimension of the orbit is the vector dimension of $T_{\beta_0} \mathcal{O}(\beta_0)$.

Now let $\beta_0 \in \mathcal{A}^n$ be a standard associative algebra and let us consider a perturbation β of β_0 . According to Goze's theorem, for an integer $p \leq n$, there exist $\epsilon_1, \dots, \epsilon_p$ infinitely small, with $\epsilon_1 \neq 0$, and $\varphi_1, \dots, \varphi_p$ independent bilinear mappings such that:

$$\beta = \beta_0 + \epsilon_1 \varphi_1 + \epsilon_1 \epsilon_2 \varphi_2 + \dots + \epsilon_1 \dots \epsilon_p \varphi_p.$$

Denoting by $\beta_1 \circ \beta_2$ the trilinear mapping defined as

$$\beta_1 \circ \beta_2(x, y, z) = \beta_1(\beta_2(x, y), z) - \beta_1(x, \beta_2(y, z)) + \beta_2(\beta_1(x, y), z) - \beta_2(x, \beta_1(y, z)),$$

β is an associative algebra if and only if $\beta \circ \beta \equiv 0$. If the infinitesimal part is denoted by ξ , then $\beta = \beta_0 + \xi$ and the latter condition is written:

$$\beta \in A^n \Leftrightarrow (\beta_0 + \xi) \circ (\beta_0 + \xi) = \beta_0 \circ \beta_0 + \beta_0 \circ \xi + \xi \circ \beta_0 + \xi \circ \xi = 0,$$

Since $\beta_0 \in \mathcal{A}^n$ and $\beta_0 \circ \xi = \xi \circ \beta_0 = \delta_{\beta_0}^2 \xi$, we have:

$$2\delta_{\beta_0}^2 \xi + \xi \circ \xi = 0, \quad (6)$$

where

$$\delta_{\beta}^2 \varphi(x, y, z) = \beta(\varphi(x, y), z) - \beta(x, \beta(y, z)) + \varphi(\beta(x, y), z) - \varphi(x, \beta(y, z))$$

are the 2-cocycles of Hochschild cohomology.

This is the perturbation equation of β_0 . Developing the expression, dividing by ϵ_1 , and separating the standard and the infinitesimal part, we obtain:

$$\begin{aligned} \delta_{\beta_0} \varphi_1 &= 0, \\ 2(\epsilon_2 \delta_{\beta_0}^2 \varphi_2 + \dots + \epsilon_2 \dots \epsilon_p \delta_{\beta_0}^2 \varphi_p + \epsilon_1 \varphi_1 \circ \varphi_1 + \epsilon_1 \epsilon_2^2 \varphi_2 \circ \varphi_2 + \\ &+ \dots + \epsilon_1 \epsilon_2^2 \dots \epsilon_p^2 \varphi_p \circ \varphi_p) + \epsilon_1 \epsilon_2 \varphi_1 \circ \varphi_2 + \dots + \epsilon_1 \dots \epsilon_p \varphi_1 \circ \varphi_p + \\ &+ \epsilon_1 \epsilon_2^2 \epsilon_3 \varphi_2 \circ \varphi_3 + \dots + \epsilon_1 \epsilon_2^2 \dots \epsilon_{p-1}^2 \epsilon_p \varphi_{p-1} \circ \varphi_p = 0. \end{aligned}$$

4 Rigid laws

The interest of determining which associative algebras are rigid lies in the main role they play in the study of the variety, because their orbits are the open components of \mathcal{A}^n . In this paragraph we calculate the dimension of the orbits of the associative algebra laws obtained in the classification theorem and prove which of them are rigid. From now on, φ_i will denote the Jordan algebras of theorem 1, and β_i the associative algebras of theorem 2.

Let $f \in GL(2, \mathbb{R})$, with $f(e_1) = ae_1 + be_2$ and $f(e_2) = ce_1 + de_2$, be a non-singular endomorphism. Evaluating (5) for each β_i , the following tangent spaces to the orbits are obtained:

$T_{\beta_1} f = \begin{pmatrix} a & b \\ -b & a \\ -b & a \\ a - 2d & b + 2c \end{pmatrix}$	$T_{\beta_2} f = \begin{pmatrix} a & b \\ b & a \\ b & a \\ 2d - a & 2c - b \end{pmatrix}$	
$T_{\beta_3} f = \begin{pmatrix} a & b \\ 0 & a \\ 0 & a \\ 0 & c \end{pmatrix}$	$T_{\beta_4} f = \begin{pmatrix} 0 & 0 \\ 0 & b + d \\ 0 & b + d \\ -c & d \end{pmatrix}$	
$T_{\beta_5} f = \begin{pmatrix} -c & 2a - d \\ 0 & c \\ 0 & c \\ 0 & 0 \end{pmatrix}$	$T_{\beta_6} f = \begin{pmatrix} a & 0 \\ 0 & a \\ c & 0 \\ 0 & c \end{pmatrix}$	$T_{\beta_7} f = \begin{pmatrix} a & 0 \\ c & 0 \\ 0 & a \\ 0 & c \end{pmatrix}$

Therefore, the dimension of the orbits are:

$$\begin{aligned} \dim \mathcal{O}(\beta_1) &= \dim \mathcal{O}(\beta_2) = 4, \\ \dim \mathcal{O}(\beta_3) &= \dim \mathcal{O}(\beta_4) = 3, \\ \dim \mathcal{O}(\beta_5) &= \dim \mathcal{O}(\beta_6) = \dim \mathcal{O}(\beta_7) = 2. \end{aligned}$$

To determine the open components of \mathcal{A}^2 , we will apply theorem 2 and some properties of the variety of Jordan algebras. In particular, we will make use of the following theorem [1]:

Theorem 4. *The only rigid algebras in \mathcal{J}^2 are φ_1 , φ_2 and φ_6 .*

Theorem 5. *The only two dimensional real associative algebras which are rigid are β_1 , β_2 , β_6 and β_7 . Thus, \mathcal{A}^2 has two open components of dimension 4 and two open components of dimension 2.*

Proof. Let $\beta_1 = \varphi_1 + \mu_1$ be, with φ_1 and β_1 the associated Jordan and Lie algebras respectively. If β is a perturbation of β_1 , then β admits a decomposition of the form $\beta = \varphi + \mu$, where $\varphi \sim \varphi_1$ and $\mu \sim \mu_1$. Since φ_1 is rigid, there exists a basis $\{e_1, e_2\}$ such that $\varphi = \varphi_1$. Then, via the classification theorem (see example 1), $\mu \equiv 0$ and β is isomorphic to β_1 . An analogous reasoning proves the rigidity of β_2 .

To prove that β_6 is rigid, let us consider a perturbation $\beta = \varphi + \mu$, with $\varphi \sim \varphi_6$. Since φ_6 is rigid, φ is isomorphic to φ_6 and it is possible to find an infinitely close basis $\{x_1, x_2\}$, where the structure constants of φ coincide with those of φ_6 and there exist ϵ and ϵ' infinitely small such that

$$\mu(x_1, x_2) = \epsilon x_1 + \left(\frac{1}{2} + \epsilon'\right)x_2,$$

According to our classification, the associativity condition imposes that $\epsilon = 0$ and $\frac{1}{2} + \epsilon' = \pm\frac{1}{2}$. Since $\frac{1}{2}$ and $-\frac{1}{2}$ are not infinitely close, $\epsilon' = 0$ and β is isomorphic to β_6 . The same reasoning proves the rigidity of the algebra β_7 .

Now let $\beta \in \mathcal{A}^2$ be a standard law non-isomorphic to any of the latter algebras. Then there exists a basis $\{e_1, e_2\}$ in which the structure constants of β are given by one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We may consider the perturbations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \epsilon & 0 \end{pmatrix}, \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \epsilon & 0 \end{pmatrix},$$

all of them without isotropy. Then, by proposition 2, these perturbations are isomorphic either to β_1 or to β_2 , and the laws β_3 , β_4 and β_5 are not rigid. \square

5 Contractions of associative algebras

Using the action of the general linear group over the variety of real associative algebras, it is possible to define a formal notion of limit in \mathcal{A}^n , in analogy with the theory of contractions developed for Lie [3] and Jordan algebras [1].

Definition 6. Let $\beta_0 \in \mathcal{A}^n$ be a real associative algebra and let $\{f_t\} \subset GL(n, \mathbb{R})$ be a family of non-singular endomorphisms depending on a continuous parametre t . If the limit

$$\beta(x, y) := \lim_{t \rightarrow 0} f_t^{-1} \circ \beta_0(f_t(x), f_t(y)) \quad (7)$$

exists for all $x, y \in \mathbb{R}^n$, β is an associative algebra called contraction of β_0 .

Example 2. β_3 is a contraction of β_1 by the linear transformations

$$f_t(e_1) = e_1, \quad f_t(e_2) = te_2.$$

Let us consider in β_1 the transformed basis $\{x_1 = f_t(e_1), x_2 = f_t(e_2)\}$, where vector products are given by:

$$\begin{aligned} x_1 \circ x_1 &= e_1 \circ e_1 = e_1 = x_1 \\ x_1 \circ x_2 &= te_1 \circ e_2 = te_2 = x_2 \\ x_2 \circ x_1 &= te_2 \circ e_1 = te_2 = x_2 \\ x_2 \circ x_2 &= t^2 e_2 \circ e_2 = -t^2 e_1 = -t^2 x_1 \end{aligned}$$

Thus, the structure constants of β_1 are represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -t^2 & 0 \end{pmatrix}.$$

It is immediate that (7) holds for e_1, e_2 . In the limit, an algebra isomorphic to β_3 is obtained. Therefore, β_3 is a contraction of β_1 .

It is easy to prove from the definition that a contraction of β_0 corresponds to a closure point of $\mathcal{O}(\beta_0)$. In particular, rigid algebras are not obtained as a contraction of any non-isomorphic algebra in \mathcal{A}^n [7]. It is obvious that the change of basis $x_i = te_i$, $i = 1, \dots, n$ induces a contraction of any associative algebra over the Abelian algebra. Moreover, for every contraction the following inequality holds:

$$\dim \mathcal{O}(\beta_0) > \dim \mathcal{O}(\beta)$$

That gives us a first criterion to study the contractions of the variety. If β is a contraction of β_0 , then its associated Jordan algebra φ is also a contraction of φ_0 . We have already proved [1] that β_4 is not a contraction of β_1 in \mathcal{J}^2 . The following theorem specifies how to obtain the remaining contractions.

Theorem 6. *Let β_i be the associative algebras of the classification theorem. Then β_3 , β_4 and β_5 are the only algebras in \mathcal{A}^2 which appear as the contraction of an associative algebra. More precisely:*

1. β_3 is a contraction of β_1 and β_2 .
2. β_4 is a contraction of β_2 .
3. β_5 is a contraction of β_1 , β_2 , β_3 and β_4

Proof. Since β_1 , β_2 , β_6 and β_7 are rigid algebras, they are not contractions of any other associative algebra.

1. We have already proved that when we consider the family of non-singular endomorphisms

$$f_t(e_1) = e_1, \quad f_t(e_2) = te_2,$$

a contraction of β_1 over β_3 is obtained. The same family of linear transformations defines the contraction of β_2 over β_3 .

2. Considering the parametric change of basis

$$f_t(e_1) = te_1, \quad f_t(e_2) = \frac{1}{2}(e_1 + e_2),$$

the limit when $t \rightarrow 0$ gives an algebra isomorphic to β_4 .

3. When considered in β_1 , the family of linear transformations

$$f_t(e_1) = \sqrt{\frac{t}{2}}(e_1 + e_2), \quad f_t(e_2) = te_2$$

defines an algebra isomorphic to β_5 in the limit.

The parametric change of basis

$$f_t(e_1) = te_2, \quad f_t(e_2) = t^2e_1$$

induce a contraction of β_2 over β_5 .

In the same way, if we take the family of non-singular endomorphisms

$$f_t(e_1) = t(e_1 + e_2) \quad f_t(e_2) = t^2e_2,$$

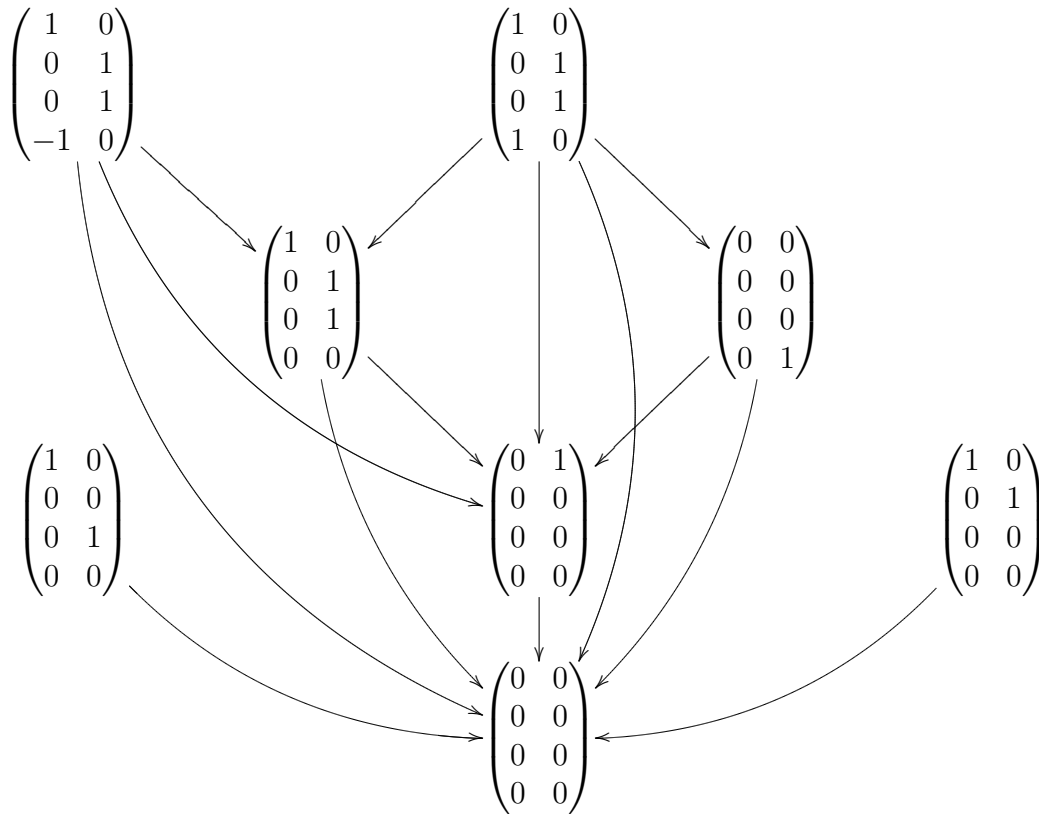
a contraction of β_3 over β_5 is obtained.

Finally, if we apply to β_4 the linear transformations

$$f_t(e_1) = e_1 + te_2, \quad f_t(e_2) = t^2e_2,$$

in the limit when $t \rightarrow 0$, it comes that β_5 is a contraction of β_4 . \square

We summarize the classification and the contractions obtained for two dimensional real associative algebras in the following diagram, where each contraction is represented by an arrow. One can verify that rigid algebras correspond to those matrices which do not receive any arrow.



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Received: May 14, 2007